Solution schemes of inhomogeneous partial differential equations using the finite difference approach

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Abstract

In this research work, we present an exhaustive investigation of a numerical algorithm for the purpose of solving the partial differential equation (PDE) in an inhomogeneous form. The proposed approach utilizes a finite difference system to numerically solve the second-order partial differential equation inherent in the inhomogeneous PDE. The proposed finite difference method proves to be highly efficient in handling linear second-order PDEs of this nature. To confirm the usefulness of our technique, we compare the approximate solution obtained using the finite difference system with the analytic solution of inhomogeneous PDE. Remarkably, the approximate numerical solution aligns satisfactorily with the purely analytic solution, demonstrating the accuracy of our proposed approach. Furthermore, we delve into the numerical solutions generated by the finite difference scheme, varying the space step size and time step size. By presenting two illustrative examples, we ascertain the usefulness and efficiency of the finite difference procedure in solving the inhomogeneous partial differential equation. Our findings contribute to the field of advanced numerical analysis and provide a valuable tool for tackling similar types of PDEs.

Keywords: inhomogeneous partial differential equation (IPDE), finite difference method (FDM), numerical examples.

1. Introduction

The inhomogeneous PDE is a fundamental mathematical equation in physics that describes the behavior of scalar particles, such as mesons, in quantum field theory. Solving this equation analytically is often challenging due to its complex nature and nonlinearity. As a result, numerical methods have become indispensable tools for obtaining solutions and gaining insights into the behavior of these systems. Among the various numerical techniques available, finite difference schemes have proven to be reliable and efficient for solving partial differential equations (PDEs) of inhomogeneous form. In present paper, we concentrate on developing a robust finite difference system for efficiently computing mathematical solutions of the partial differential equation of inhomogeneous form. The finite difference method discretizes the continuous inhomogeneous partial differential equation on a grid, approximating the derivatives with finite difference approximations. As a result of discretization, the PDE is transformed into a series of algebras that can be solved numerically. By carefully choosing the grid spacing and time steps, we can strike a balance between accuracy and computational efficiency. The principal objective of this research work is to investigate the proposed finite difference scheme and evaluate its effectiveness in accurately solving the inhomogeneous partial differential equation. We aim to demonstrate that the scheme is not only efficient but also capable of producing satisfactory approximations when compared to known analytic solutions. To accomplish this, we will first derive the discretized form of the inhomogeneous PDE employing the scheme of finite difference. We will then examine the approximations of the numerical solutions to the corresponding analytical solutions to properly evaluate the accuracy of the suggested method, whenever possible. Additionally, we will investigate the sensitivity of the results to variations in the spatial (or space) step size size and temporal (or time) step size, enabling us to assess the consistency and robustness of the method of finite difference. Islam *et al.* (2018a, 2018b) discussed

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numerical solutions of wave and heat equations using finite difference method. Islam and Karim (2019) introduced Crank–Nicolson finite difference scheme for solving convection-diffusion equation. Feng and Li (2013) discussed the stability of a one-dimensional wave equation of partial differential equation. Wazwaz (1998) presented decomposition method to find the solution of one-dimensional wave equation. Chun et al. (2009) discussed homotopy perturbation method for obtain the solution of different type of partial differential equations. Szyszka (2017) presented finite difference method to solve one-dimensional wave equation. He (2005) discussed solution procedure of partial differential equations (PDE) applying homotopy perturbation method. Abbasbandy (2008) analyzed a numerical method to solve different type of partial differential equations. Noor and Mohyud-Din (2008) discussed solution of nonlinear partial differential equation of higher order employing variational iteration method. Han *et al.* (2005) introduced a finite-difference method to solve Schrödinger equation of one-dimensional.

In this research, we used the approach of finite difference to solve inhomogeneous partial differential equations (IPDEs). Overall, this paper aims to contribute to the area of advanced numerical analysis by providing an efficient and reliable method to solve the inhomogeneous partial differential equation (IPDE). To validate the proposed approach, we have presented two illustrative examples that highlight the versatility and reliability of the finite difference system in solving the inhomogeneous partial differential equations (IPDEs).

2. Finite difference scheme of inhomogeneous partial differential equation (IPDE)

In this study, we investigate the linear partial differential equation (PDE) of the inhomogeneous form, along with its associated initial and boundary conditions (IBC). We employ the finite difference approach, a highly effective numerical technique based on mathematical discretization, to obtain computational solutions for this differential equation. The finite difference schemes are obtained by truncating the Taylor's series method, leading to a combination of algebraic equations that replace the original PDE and initial-boundary conditions. The finite difference method has proven to be a versatile approach widely employed in engineering to approximate solutions for partial differential equations encountered in diverse fields such as heat transfer, biological system, dynamical system and fluid dynamics etc. Our primary objective in this research is to determine accurate approximate computational solutions of IPDE for boundary value problem (BVP) formulated by equation (2.1) using the method of finite difference. We address the general form of inhomogeneous PDE to demonstrate the finite difference method.

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} + g(x) \tag{2.1}$$

The provided partial differential equation (PDE) of inhomogeneous form can be characterized as:

$$w_t = a^2 w_{xx} + g(x) \tag{2.2}$$

with initial-boundary conditions (IBC):

$$\left.\begin{array}{l} w(0,t)=0,\ w(r,t)=0;\ t>0 \\ w(x,0)=f(x);\ 0<x<r \end{array}\right\} \tag{2.3}$$

The finite difference approximations are given by

$$\frac{\partial w}{\partial t} \approx \frac{w(i,j+1)-w(i,j)}{k} \tag{2.4}$$

$$\frac{\partial^2 w}{\partial x^2} \approx \frac{w(i+1,j+1)-2w(i,j+1)+w(i-1,j+1)+w(i+1,j)-2w(i,j)+w(i-1,j)}{2h^2} \tag{2.5}$$

We get the following equation by substituting (2.4) and (2.5) in eqn. (2.1):

$$\frac{w(i,j+1)-w(i,j)}{k}=a^2\,\frac{w(i+1,j+1)-2w(i,j+1)+w(i-1,j+1)+w(i+1,j)-2w(i,j)+w(i-1,j)}{2h^2}+g(x_i)$$

$$w(i,j+1)-w(i,j)=\frac{a^2k}{2h^2}\{w(i+1,j+1)-2w(i,+1)+w(i-1,j+1)+w(i+1,j)-2w(i,j)+w(i-1,j)\}+kg(x_i)$$

…….(2.6)

For an inhomogeneous partial differential equation, equation (2.6) is known as the finite difference scheme. This equation serves as a foundation for understanding the methodology employed and demonstrates the potential applications of the finite difference method (FDM) in this context. By applying the FDM to the given problem, we aim to obtain close approximations of the computational solution that satisfy the prescribed initial and boundary points. Through this research, we aim to contribute to the understanding and utilization of the finite difference method in solving PDEs, specifically focusing on the inhomogeneous partial differential equation. By successfully achieving accurate approximate solutions, we enhance our comprehension of the dynamics and behavior described by this equation, paving the way for further applications in relevant scientific and engineering domains.

3. Numerical Solution Procedure

In the specified domain of inhomogeneous partial PDE, the finite difference approach is the most crucial computational numerical method utilized to solve real-world engineering problems. It is essential to emphasize that the stability and accuracy of the suggested procedure depend on the choice of Δx and Δt. The finite difference process is applied to approximate derivatives that are partial at a point (x_i, t_j) by discretizing the two-dimensional plane into a grid of rectangular cells, where $\Delta x = h$ represents the length and $\Delta t = k$ denotes the width of each cell, using lines $x_i = i\Delta x = ih$ and $t_j = j\Delta t = jk$ that the x and y axes are parallel. Mesh points are the points at which these lines intersect. The mesh points (x_i, t_j) are represented by $w(i, j)$. The equation (2.6) can be used to determine the approximate solution of $w(i, j)$ for all possible i and j values. In this finite difference scheme has truncation error of spatial order $O((\Delta x)^2) = O(h^2)$ and temporal order $O((\Delta t)) = O(k)$.

4. Numerical Examples

To assess the efficacy of the suggested FDM, we investigate two different examples of inhomogeneous partial differential equations. Figures 4.1(a)-4.1(d) and 4.2(a)-4.2(d) graphically depict the approximate results of the solutions.

Example 4.1: we analyse a one-dimensional inhomogeneous partial differential equation (IPDE):

$\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}+\sin(x)$; $w(x,0)=\cos x,\ w(0,t)=e^{-t},\ w(\pi,t)=-e^{t};\ 0<x<\pi,\ t\geq 0.$ The accurate solution is provided by $w(x,t)=e^{-t}\cos x+(1-e^{-t})\sin x$. Figure 4.1(a) illustrates the precise solution, whereas Figures 4.1(b)–4.1(d) depict the graphs of the numerical solutions to the inhomogeneous partial differential equation.

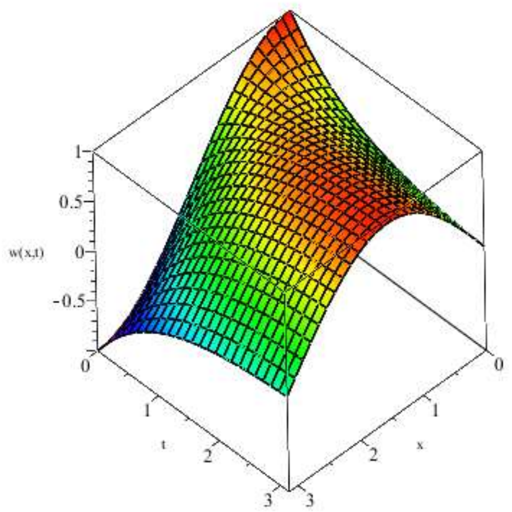

Fig.4.1(a): the exact numerical solution $w(x,t)$ for various values of *x* and *t*.

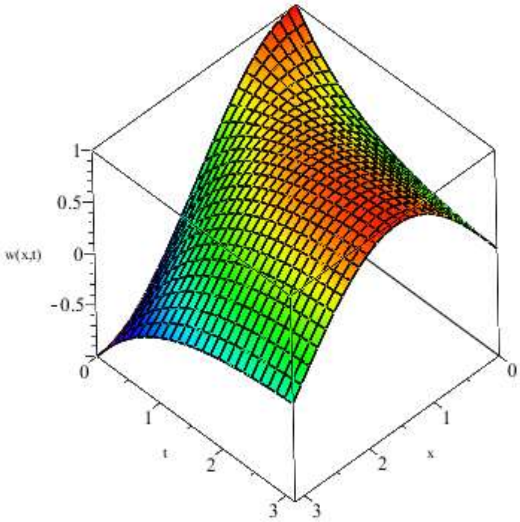

Fig. 4.1(b): the approximate numerical solution $w(x,t)$ for *h*=0.10 and *k*=0.10.

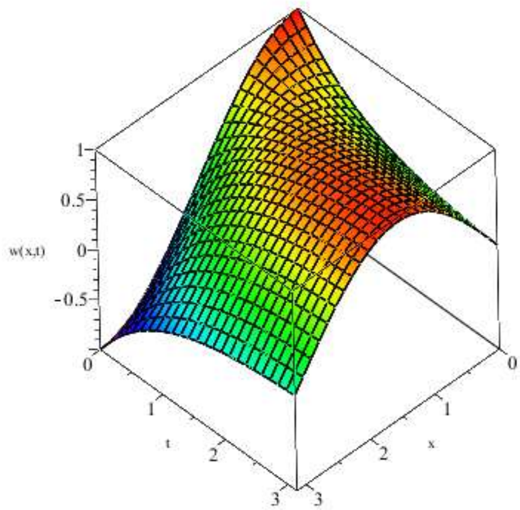

Fig.4.1(c): the approximate numerical solution $w(x,t)$ for *h*=0.05 and k=0.05.

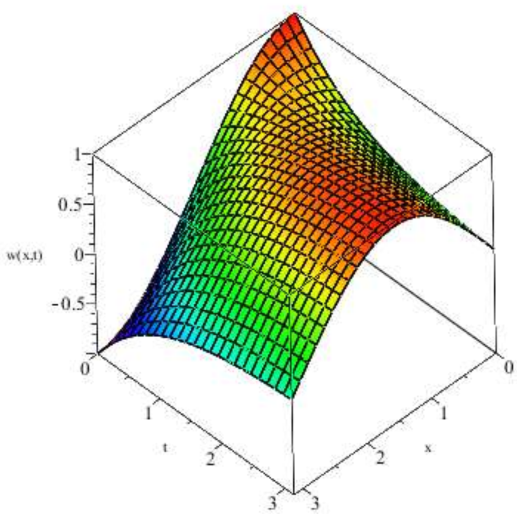

Fig.4.1 (d): the approximate numerical solution $w(x,t)$ for *h*=0.001 and *k*=0.001.

Example 4.2: we analyse an inhomogeneous partial differential equation (IPDE):

$\dfrac{\partial w}{\partial t}=\dfrac{\partial^2 w}{\partial x^2}+\cos(x)$; $w(x,0)=0,\ w(0,t)=1-e^{-t},\ w(\pi,t)=e^{-t}-1;\ 0<x<\pi, t\geq 0.$ The accurate solution is provided by $w(x,t)=(1-e^{-t})\cos(x)$. Figure 4.2(a) illustrates the precise solution, whereas Figures 4.2(b)–4.2(d) depict the graphs of the numerical solutions to the inhomogeneous partial differential equation.

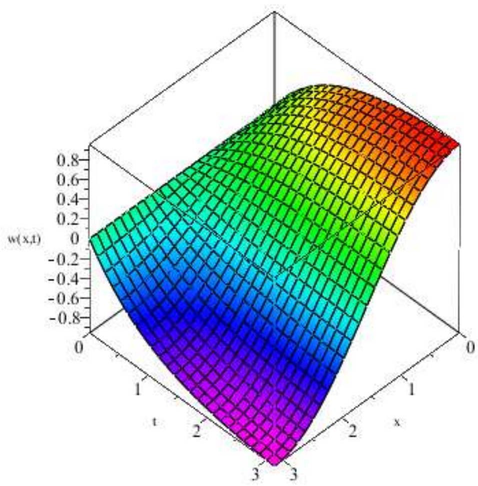

Fig. 2(a): the exact numerical solution $w(x,t)$ for various values of *x* and *t*.

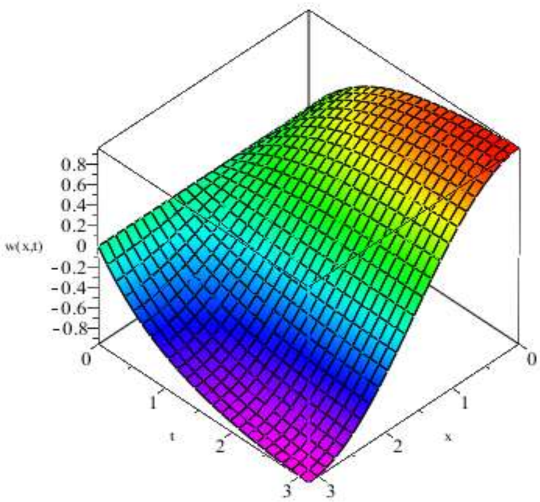

Fig.2 (b): the approximate numerical solution $w(x,t)$ for *h*=0.10 and *k*=0.10.

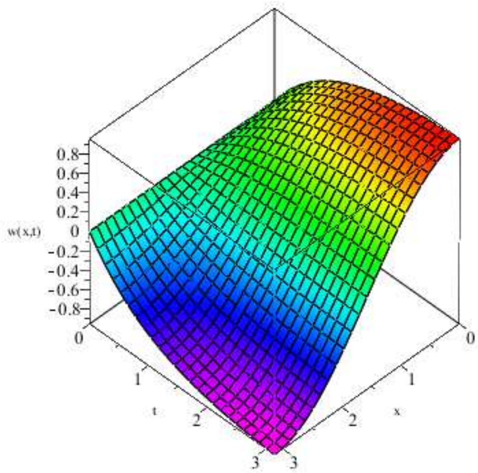

Fig. 2 (c): the approximate numerical solution $w(x,t)$ for *h*=0.05 and k=0.05.

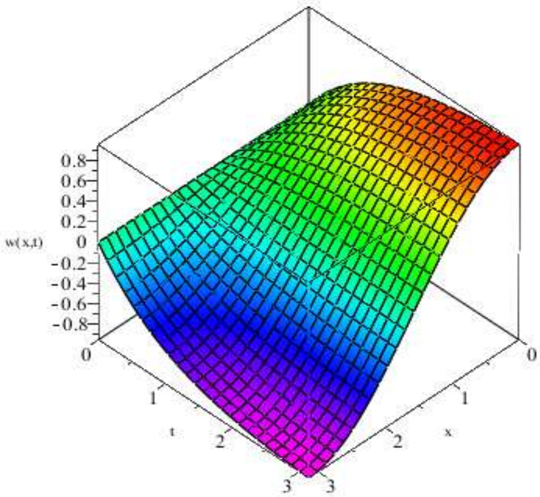

Fig.2 (d): the approximate numerical solution $w(x,t)$ for *h*=0.001 and *k*=0.001.

5. Discussion of the results

As shown by comparing the estimated computational results with the true numerical solutions using two practical examples of an inhomogeneous partial differential equation, the exactness of the numerical solution greatly depends on the selected step size. Figures 4.1(b)-4.1(d) and 4.2(b)-4.2(d) show the effects of spatial discretization (h) and temporal discretization (k) on the numerical solution, respectively. Both h and k should be set to small numbers for greater precision, with k being very small. When the approximate solution gets closer to the precise solution, a numerical procedure is deemed convergent.i.e., $\left|w_{exact}(x_i,t_j)-w_{app}(x_i,t_j)\right|=0 \quad \text{for}\, h\to 0, k\to 0$ Where $w_{exact}(x_i,t_j)$ represents the exact computational numerical result and $w_{app}(x_i,t_j)$ symbolizes the approximate numerical result. The proposed finite difference method (FDM) is the most reliable method for obtaining an approximate mathematical solution to the inhomogeneous partial differential equation because it exhibits faster convergence when both h and k are minimized, according to an analysis of the approximated solution performed using the Maple software across various step sizes.

6. Conclusion

In this research, we have successfully developed and implemented a numerical solution technique for inhomogeneous partial differential equations (IPDE) using the finite difference approach. We provide two comprehensive numerical experiments to test the effectiveness and reliability of the suggested numerical method. For this proposed method to produce more precise results, the mesh size and time interval must be reduced. The figures make it clear that the precision of the problem's solution depends on the mesh size (h) and time interval (k). The computational solutions derived from the proposed finite difference method for inhomogeneous partial differential equations exhibit a strong correlation with the exact numerical solutions. The results obtained through this method are generally characterized by higher accuracy of the estimated solution towards the exact numerical solution is notably faster.

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