A Source Pack Pairing Method for Formation of Magic Squares of Order 3

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Abstract

In this article we have developed a method by which innumerable magic squares of order 3 can be constructed. We have cited several interesting examples for construction of such magic squares. The core idea is that if we assume an arbitrary source pack comprising of 9 distinct integers in the form of an arithmetic progression then by using this method, we can construct 8 different magic squares of order 3 for each source pack. The description of the method has been made elaborate only for reasons of clarity. We have stated and proved two conjectures which play vital roles in the development of the process.

Keywords: array of numbers, source pack, sequence of integers, middle number, magic sum, main diagonal and minor diagonal, corner positions.

1. Introduction

A magic square is a sort of mathematical recreation practiced by cross section of people in most regions around the globe over many countries. It is a game of arranging some integers in a square array with the goal of making the sum of the numbers along the rows or the columns or the diagonals of the array the same constant value.

The first record of a magic square is found in China.[1], [3], [5] Legend has it that around 190 B.C some fisher men of the river Lo brought a turtle to the great Chinese emperor Yii who was also a hydraulic engineer. The hard upper surface of the turtle was engraved with the following chart:

4	9	2
3	5	7
8	1	6

The emperor recognized the magic square of order 3 and became immediately interested in it.
Since then magic squares have been leisurely practiced in China, India, Arabia and Persia. In most of these regions, magic squares were related to the positions of sun, moon, planets and other astronomical objects. In India magic squares were also related to religious mythologies**.** [8], [10]. Europe got the essence of magic square late, towards the end of 15th century through translation of Arabic books. But they began to develop the subject on scientific lines

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and found applications in making patterns and art works.

Celebrated artist Albert Durer's masterpiece engraving "*Melacholia*" (kept in the British Museum) has a magic square of order 4 prominently displayed on the great art work. In recent years, however, the most organized and analytic works in magic squares are done in several universities of the U.S.A. [6], [7]. Although the statement of the magic square problem is simple, but the solution of the problem does not see to be that easy. Over the centuries, only some adhoc solutions are available for magic squares of order less than or equal to 5. The quest for general rules remains a challenge as ever. [2], [3]

In this article we describe a process by which magic squares of order 3 are obtained from a set of 9 distinct numbers called the "Source Pack" and of course, the efforts to generalize the process shall remain open as ever.

2. Description of the problem and some related terms

A magic square of order n is an array of $n \times n$ distinct integers arranged in such a way that the sum of the numbers in each row, in each column or in each diagonal is the same constant number. This is the basic magic square hypothesis.

A magic square of order 3, have 9 distinct integers which must be chosen from a set of 9 integers which we call the "Source Pack" σ .The distinct 9 numbers of the source pack for a problem can be arranged in one or more ways such that the magic square hypothesis is satisfied.

Obviously, the elements of the source pack cannot be completely unrelated numbers, because together they have to satisfy the magic square hypothesis. We assert that the elements of a source pack must be a sequence of integers forming an arithmetic progression:

$$\sigma = \{a, a+d, a+2d, \ldots\ldots(c), \ldots\ldots, (l-d), l\}$$

where a is the first term, l the last term, d the common difference and c is the central (middle) number.

For a source pack of 9 elements, we have,

$$\sigma = \{a, a+d, a+2d, a+3d, (a+4d), a+5d, a+6d, a+7d, a+8d\} \ldots\ldots\ldots\ldots(1)$$

Where the middle number is $(a+4d)$ and the last number is $(a+8d)$.

If the source pack contains the first 9 consecutive integers,

$$\sigma = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \ldots\ldots\ldots\ldots(2)$$

then σ is called the **'Basic or the Normal Source Pack'** and the magic squares produced by it are called the Basic or the Normal magic squares of order 3.

In addition to the basic source pack, we shall use several other source packs which are all some arithmetic progressions.

3. Pairing of the elements of a source pack

For construction of a magic square of order 3, first we fix the middle number c at the central position a_{22} of the array. Then we make pair of numbers equidistant from the central number c on both sides of it:

$\sigma = \{a, a+d, a+2d, a+3d, (c = a+4d), a+5d, a+6d, a+7d, a+8d\}$ So that the pairs are:

$P_1 = (a, a+8d)$, the 1st term and the last term.

$P_2 = (a+d, a+7d)$, the 2nd term and the 8th term.

$P_3 = (a+2d, a+6d)$, the 3rd term and the 7th term.

$P_4 = (a+3d, a+5d)$, the 4th term and the 6th term.

The 5th term $c = a+4d$ is the middle number of the source pack.

The statement that we place the pair P_1 at a corner means that any one element of P_1 (say, a) is placed at the corner positions a_{11} or a_{13} or a_{31} or a_{33} and the other element $(a+8d)$ of P_1 is placed at the diagonally opposite position a_{33} or a_{31} or a_{13} or a_{11} respectively.

Also, if P_2 is placed along the second column then it's one element $(a+d)$ is placed at the position a_{12} and its other element $(a+7d)$ is placed at a_{32} position.

Similar rule goes for rows: if $(a+3d)$ of P_4 is placed at a_{21} then the other element $(a+5d)$ of P_4 is to be placed at a_{23}.

In short, elements of a pair always occupy reflective positions, the end position of a row a or a column or diagonal. Finally, after filling the central position and the corner positions, there will remain exactly 4 vacant positions. These vacant positions must be filled in by inspection, so that the sum of the numbers of each column or row or diagonal add up to the magic number. The numbers of the vacant positions shall always come from the elements of the unused pairs as a consequence of the magic square hypothesis.

4. Determination of the magic sum

Given a source pack $\sigma = \{a,\ a+d,\ \ldots\ldots\ldots,\ (a+4d),\ \ldots\ldots\ldots\ldots,a+8d\}$, it is easy to compute the magic number for the 3×3 magic square. Let the sum of the numbers of the source pack be S.

Then $S = a + (\ a+d) + \ldots\ldots\ldots + (a+4d) + \ldots\ldots\ldots\ldots + (a+8d).$

Reversing the order of the addends, we can write,

$S = (\ a+8d) + (a+7d) + \ldots\ldots\ldots + (a+4d) + \ldots\ldots\ldots\ldots + a.$

Adding we get $2S = (\ 2a+8d) + (2a+8d) + \ldots\ldots\ldots;$ (9 terms)

$$\therefore S = \frac{9\ (2a+8d)}{2} \quad = 9\ (a+4d).$$

This sum can be thought of as the sum of 3 rows (or 3 columns). Therefore, the magic sum is

$$m = \frac{9\ (a+4d)}{3} \quad = 3\ (a+4d).$$

For the basic source pack $\sigma = \{1, 2, 3,4, 5, 6, 7, 8, 9\},$ the magic number is $m = 3\times 5 = 15.$

Thus, for any source pack σ ,we can readily compute its magic number m , because it depends on the contents of σ .

Now we state and prove two conjectures stating that the pairs $P_1 = (a, a+8d)$ and $P_3 = (a+2d, a+6d)$ cannot be placed at any corner position.

5. Conjecture-1

For construction of a magic square of order 3, assuming that the middle number $(a+4d)$ is placed at the central a_{22} position, the elements of the pair $P_1 = (a,\ a+8d)$ cannot be used as entries of the corner positions.

Proof: Consider a source pack $\sigma = \{a, a+d,\ \ldots\ldots\ldots, (a+4d), \ldots\ldots\ldots\ldots,a+8d\}$ of which the middle number $(a+4d)$ is placed at the central position and the pairings are:

$P_1 = (a,\ a+8d)$, $P_2 = (a+d,\ a+7d)$, $P_3 = (a+2d, a+6d)$, $P_4 = (a+3d, a+d)$,

The magic sum for σ is $m = 3\ (a+4d).$

If possible, suppose P_1 is placed at the corners along the main diagonal and the middle number $(a+4d)$ at the central a_{22} position and P_2 along the minor diagonal producing a situation of table.1.

a		$a+d$
	$a+4d$	
$a+7d$		$a+8d$

Table: 1

We can see that at row 3, the magic sum criterion is violated.
By reversing the positions of the elements of P_1 and P_2 one by one magic sum criterion is violated at different row or column.
Therefore, pairs P_1 and P_2 cannot occupy the corner positions simultaneously.
Now we try P_1 and P_3 pairs at the corner positions and find that the magic number criterion is violated at some other row or column. Therefore P_1 and P_3 cannot occupy the corner positions simultaneously.
In a similar way, P_1 and P_4 also cannot occupy corner positions simultaneously.
Hence, we conclude that the elements of the pair P_1 cannot be placed at any of the corner positions.

6. Conjecture-2

For construction of a magic square of order 3 and assuming that the middle number $(a+4d)$ is placed at the central a_{22} position, the elements of the pair $P_3=(a+2d, a+6d)$ cannot be used as entries of the corner positions.
Proof: The magic sum of the source pack is $m=3a+12d$. We have already proved that P_3 and P_1 cannot be placed at the corner positions simultaneously.
We now test if P_3, P_2 or P_3, P_4 pairs can fill the corner positions. Consider P_3 along the main diagonal and P_2 along the minor diagonal and refer to table-2:

$a+2d$		$a+d$
	$a+4d$	
$a+7d$		$a+6d$

Table2: (P_3 and P_2 at corners)

$a+2d$		$a+3d$
	$a+4d$	
$a+5d$		$a+6d$

Table3: (P_3 and P_4 at corners)

Whatever elements fill the vacant position, the magic sum criterion is violated in row 3.
By reversing the positions of elements P_3 and then those of P_2 we see that the magic sum criterion is violated at different row or column. Therefore, P_3 and P_2 cannot occupy corner positions simultaneously. For the case P_3 and P_4 at the corner positions, they do not violate magic number criterion as yet, but the vacant positions must be filled in by elements of P_1 and P_2 .Whatever numbers from P_1 and P_2 are given, at least one row or column violate the magic number criterion.
By reversing positions and by interchanging major and minor diagonals, at least one row or column violate the

magic number criterion.

Therefore, the elements of the pair P_3 cannot be used as entries in any of the corner points.

7. Construction of 3×3 magic squares

Because of the conjectures 1 and 2, we are now in a position to construct innumerable number of 3×3 magic squares. We take any source pack consisting of 9 integers in a sequence, fix the middle number at the central position a_{22} and form the pairs P_1, P_2, P_3, P_4 which are equidistant pairs relative to the middle number.

Since P_1 and P_3 cannot be placed at the corner positions so we put elements of P_2 and P_4 at the corner points along the main diagonal and the minor diagonal. The process will produce 4 vacant positions which must be filled in by numbers from the pairs P_1 and P_3.

The numbers are suitably chosen so that the magic number criterion is satisfied. For any source pack of 9 elements, there can be 8 different magic squares of order 3.

8. The Normal Magic Square of Order 3

Consider the normal source pack of the first 9 consecutive integers: $\sigma = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Any 3×3 magic square formed from this source pack is called a normal magic square of order 3.

Here, the middle number 5 is placed at the central a_{22} position.

The sum of numbers of the source pack is $S = \frac{9\times 10}{2} = 45$, which the sum of three rows is.

The magic sum $m = \frac{45}{3} = 15$.

We form equidistant pairs relative to the middle number,

$P_1 = (1, 9)$, $P_2 = (2, 8)$, $P_3 = (3, 7)$ and $P_4 = (4, 6)$

Since elements of P_1 and P_3 cannot be at the corner positions, we put those of P_2 and P_4 along the diagonals.

2		4
	5	
6		8

2		6
	5	
4		8

8		4
	5	
6		2

8		6
	5	
4		2

(P_2 along main diagonal and P_4 along minor diagonal)

The vacant positions are filled in such a way that the sum of any row or column is 15. That will use up the elements of P_1 and P_3. Again,

4		2
	5	
8		6

4		8
	5	
2		6

6		2
	5	
8		4

6		8
	5	
2		4

(P_4 along the main diagonal and P_2 along the minor diagonal)

Vacant positions must be filled up in such a way that the sum of a row or column is 15.

9. A 3×3 magic square of even integers

We consider a source pack containing a sequence of 9 even integers: $\sigma = \{8, 10, 12, 14, (16), 18, 20, 22, 24\}$.

The middle number is 16, which we put at the central position a_{22} and form pair of numbers equidistant from 16 on both sides:

P_1 = (8, 24), P_2 = (10, 22), P_3 = (12, 20) and P_4 = (14, 18)

The sum of numbers of the source pack is

$S = \frac{(8+24)\times 9}{2} = 144$, which is considered as sum of the numbers of 3 rows.

Therefore, the magic sum is: $m = \frac{144}{3} = 48$.

We assert that the corner positions can be filled only by the elements of P_2 and P_4. Let us put P_2 along the main diagonal and P_4 along the minor diagonal and reverse the positions by turn:

10		14
	16	
18		22

10		18
	16	
14		22

22		14
	16	
18		10

22		18
	16	
14		10

(P_2 along main diagonal, P_4 along minor diagonal with reversal of positions)

The vacant positions must be filled up by such numbers as to make the sum of a row or a column equal to 48.

Again, we put P_4 along the main diagonal and P_2 along the minor diagonal and reverse the positions in turn:

14		10
	16	
22		18

14		22
	16	
10		18

18		22
	16	
10		14

18		10
	16	
22		14

(P_4 along the main diagonal, P_2 along the minor diagonal with reversal of positions)

The vacant positions must be filled up with such numbers as to make the sum of a row or a column equal to 48.

The result is that we obtain 8 different magic squares consisting of the source pack.

10. A 3×3 magic square composed of Negative Integers, Zero and Positive Integers.

In some published articles, it appears that the entries of a magic square are composed of only positive integers. But that is not the case. We present here some magic squares which contain positive integers, zero and negative integers.

Consider the source pack $\sigma = \{-21, -14, -7, 0, (7), 14, 21, 28, 35\}$.

The middle number of the source pack is 7 and we form the pairs whose elements are equidistant on each side of the middle number:

P_1 = (-21, 35), P_2 = (-14, 28), P_3 = (-7, 21) and P_4 = (0, 14)

The sum of the numbers of the source pack is $S = \frac{(35-21)\times 9}{2} = 63.$ which is the sum of three rows of the magic square. Therefore, the magic sum, $m = \frac{63}{3} = 21$ which is the sum of any one row or one column or one diagonal of the magic square. As the principles developed in this article, the central position is occupied by the middle number 7 and the corner positions are occupied by the elements of P_2 and P_4 . First, we put P_2 along the main diagonal and P_4 along the minor diagonal and reverse the positions by turn.

-14		0
	7	
14		28

-14		14
	7	
0		28

28		0
	7	
14		-14

28		14
	7	
0		-14

(P_2 along main diagonal and P_4 along the minor diagonal with reversal of positions)

The vacant positions must be filled with numbers such that the sum of each row or column becomes 21.

Next, we put P_4 along the main diagonal and P_2 along the minor diagonal and reverse the positions in turn. In all cases the middle number 7 remains fixed at the central position a_{22}.

0		-14
	7	
28		14

0		28
	7	
-14		14

14		28
	7	
-14		0

14		-14
	7	
28		0

(the elements of P_4 along main diagonal and those of P_2 along the minor diagonal).

The remaining vacant places are to be filled by such numbers as to make the sum of each row or column equal to 21.

11. The zero-sum magic square of order 3

We now consider a source pack of 9 distinct integers of which the middle number is zero and along the number line, on the left of 0 there are four negative integers and on the right of 0 lies four equidistant positive integers.
Let $\sigma = \{-16, -12, -8, -4, (0), 4, 8, 12, 16\}$.
Sum of the element of the source pack is:
$S = -16 - 12 - 8 - 4 + 0 + 4 + 8 + 12 + 16 = 0.$
This sum is spread in the three rows and hence, the magic sum is: $m = \frac{0}{3} = 0.$
Therefore, the magic square of order 3 that we are going to construct is called the zero-sum magic square of order 3.
We form pairs of numbers equidistant from 0:
P_1 = (-16, 16), P_2 = (-12, 12), P_3 = (-8, 8) and P_4 = (-4, 4)
To form the zero-sum magic square, we place the middle number 0 at the central position a_{22}. Then we place P_2 along the main diagonal and P_4 along the minor diagonal with reversal of positions:

-12		-4
	0	
4		12

-12		4
	0	
-4		12

12		-4
	0	
4		-12

12		4
	0	
-4		-12

(P_2 along the main diagonal and P_4 along the minor diagonal)

The vacant positions must be filled with suitably chosen numbers such that the sum of any row or column adds up to 0.
Similarly, we can again put elements of P_4 along the main diagonal and those of P_2 along the minor diagonal, keeping the middle number 0 at the central position a_{22}, with reversal of positions:

-4		-12
	0	
12		4

-4		12
	0	
-12		4

4		-12
	0	
12		4

4		12
	0	
-12		-4

(P_4 along the main diagonal and P_2 along the minor diagonal).

The vacant positions must be filled up with numbers such that the sum of each row or (column) is 0.
Thus, we obtain 8 different zero-sum magic squares of order 3.
By taking different source pack, it is possible to form infinite number of zero-sum magic squares of order3.

12. Conclusion

In each of the problems discussed above, we have kept 4 positions vacant, two along a row and two along a column so that the reader participates with the construction as he reads it.

In all the examples cited in this paper, we have shown that given any source pack of 9 distinct integers in the form of an arithmetic progression, we are able to construct 8 different magic sources of order 3. By considering different source packs it is possible to form an infinite number of magic squares of order 3. However, variation of rules and construction of magic squares of order 4 or 5 or more remains open challenge and is worth of investigations. [5], [7].

Dedication

We do most respectfully dedicate this humble work to the memory of

1) Late Professor Dr. Mushfiqur Rahman and
2) Late Professor Azizur Rahman Khalifa,

who were loving teachers of the second author and very distinguished teachers of Dhaka University and eminent pioneer mathematicians of this land.

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